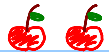
§ 2 Natural Numbers

Me : What is 1 ?

Mom: 

Me : What is 2 ?

Mom: 

Me : Why $1+1=2$?

Mom:  +  =  

Set-theoretical definition of $\mathbb{N}$:

Idea :

Definition 2.1

Given a set x , define $x^+ = x \cup \{x\}$ called successor of x .

$x^+ = x \cup \{x\}$

$(x^+)^+ = x^+ \cup \{x^+\} = x \cup \{x\} \cup \{x, \{x\}\}$

How to understand x^+ ?

$x^+ = x \cup \{x\}$ = taking the set x and adding an extra element x

Therefore, $x \subseteq x^+$ and $x \in x^+$!

Definition 2.2

A set S is said to be a successor set if and only if

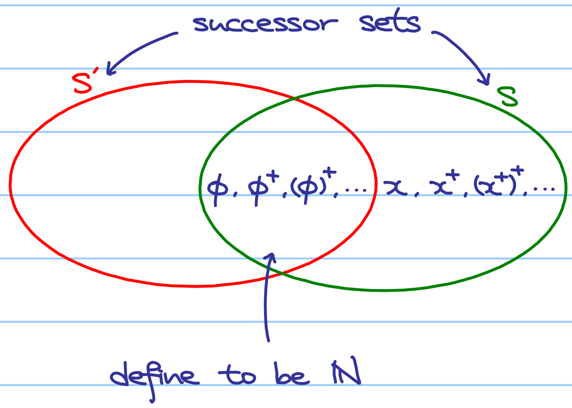
$\phi \in S$ and for all x, if $x \in S$, then $x^+ \in S$.

(Axiom of infinity guarantees the existence of a successor set.)

Proposition 2.1

There exists a unique successor set that is a subset of every successor set.

$x \in S \Rightarrow x^+ \in S \Rightarrow (x^+)^+ \in S \Rightarrow \cdots$



The natural numbers are defined recursively by letting

$0 = \phi = \{\ \}$

$1 = 0^+ = 0 \cup \{0\} = \phi \cup \{\phi\} = \{\phi\} = \{0\}$

$2 = 1^+ = 1 \cup \{1\} = \{\phi\} \cup \{\{\phi\}\} = \{\phi, \{\phi\}\} = \{0, 1\}$

$3 = 2^+ = 2 \cup \{2\} = \{\phi, \{\phi\}\} \cup \{\{\phi, \{\phi\}\}\} = \{\phi, \{\phi\}, \{\phi, \{\phi\}\}\} = \{0, 1, 2\}$

$\vdots \qquad \vdots$

$n+1 = n^+ = n \cup \{n\} = \{0, 1, 2, \cdots, n\}$

- $2 \in 3 = \{0, 1, 2\}$
- $\{0, 1\} = 2 \subseteq 3 = \{0, 1, 2\}$

Giving the symbols 1, 2, 3, … some meaning as sets.

Every natural number is defined as a set which contains all "preceding" natural numbers.

We haven't define what "≤" and "<" are, but here is something we can observe:

$m = \{0, 1, 2, \cdots, m-1\}$, $n = \{0, 1, 2, \cdots, n-1\}$

1) m "≤" n if $m \subseteq n$ (and we know $n \leq n$ and then we should have $n \subseteq n$!)

2) m "<" n if $m \in n$ (but we know $n \nless n$ and then we should have $n \notin n$!)

It motivates us how to define "≤" and "<" (Discuss later!)

Besides giving a definition of ℕ, we have to show our model has the properties or structures (e.g. $+, \times, \leq, <$) as we expected!

Theorem 2.1

1) $0 \in \mathbb{N}$

2) If $n \in \mathbb{N}$, then there exists a unique natural number n^+ which is the successor of n.

Idea: Do not allow

$$\begin{tikzcd} m=n \arrow[r, bend left] \arrow[dr, bend right] & n^+ \arrow[d, phantom, "\nparallel"] \\ & m^+ \end{tikzcd}$$

3) For all $n \in \mathbb{N}$, $n^+ \neq 0$.

Idea: Do not allow

$$\begin{tikzcd} 0 \arrow[r, bend left] & 1 \arrow[r, bend left] & 2 \cdots n \arrow[ll, bend left] \end{tikzcd}$$

4) If $n^+ = m^+$, then $n = m$.

Idea: Do not allow

$$\begin{tikzcd} m \arrow[dr] \arrow[d, phantom, "\nparallel"] & \\ n \arrow[r] & m^+ = n^+ \end{tikzcd}$$

5) (Mathematical Induction) If $S \subseteq \mathbb{N}$ such that

(i) $0 \in \mathbb{N}$ and (ii) if $n \in S$, then $n^+ \in S$

then $S = \mathbb{N}$.

(1) – (3), (5) are direct consequences of definition of $\mathbb{N}$

proof of (4) relies on the following lemma.

Lemma 2.1

For each natural number n,

1) n is a set of natural number.

2) every element of n is also a subset of n, and

3) $n \notin n$.

proof:

Applying mathematical induction and let S be the subset of $\mathbb{N}$ whose elements satisfy (1)

(i) $0 = \phi \in S$ (ϕ contains no element which is not a natural number)

(ii) If $n \in S$, n is a set of natural numbers, so $n^+ = n \cup \{n\}$ is also a set of natural numbers, i.e. $n^+ \in S$.

$\therefore S = \mathbb{N}$.

Let T be the subset of $\mathbb{N}$ whose elements satisfy (2).

(i) $0 = \phi \in T$

(ii) If $n \in T$, if $x \in n^+ = n \cup \{n\}$, then $x \in n$ or $x \in \{n\}$,

case 1: $x \in n$, then $x \overset{\text{assumption}}{\subseteq} n \subseteq n \cup \{n\} = n^+$

case 2: $x \in \{n\}$, then $x = n \subseteq n \cup \{n\} = n^+$

$\therefore\ T = \mathbb{N}$.

Let U be the subset of $\mathbb{N}$ whose elements satisfy (3).

(i) $0 = \phi \in U$

(ii) If $n \in U$, i.e $n \notin n$

Suppose $n^+ \in n^+ = n \cup \{n\}$,

case 1: $n^+ \in n$, then $n \cup \{n\} = n^+ \overset{\text{by (2)}}{\subseteq} n$ (Contradiction)

case 2: $n^+ \in \{n\}$, then $n^+ = n$ and so $n^+ = n \overset{\text{assumption}}{\notin} n = n^+$ (Contradiction)

$\therefore\ n^+ \notin n^+$, i.e. $n^+ \in U$

$\therefore\ U = \mathbb{N}$.

proof of (4) of theorem 2.1 :

Suppose $m^+ = n^+$, we have $m \in m \cup \{m\} = m^+ = n^+ = n \cup \{n\}$. Similarly $n \in m \cup \{m\}$.

case 1: $m = n$

case 2: $m \in n$ and $n \in m$, by lemma 2.1, $m \subseteq n$ and $n \subseteq m$ $\therefore\ m = n$

Proposition 2.2

1) If $n \in \mathbb{N}$, then $n \neq n^+$.

2) If $n \in \mathbb{N}$ and $n \neq 0$, then there exists a unique $m \in \mathbb{N}$ such that $n = m^+$,

in this case, we write $n = m+1$ and $m = n-1$.

proof :

1) Since $n \notin n$ for all $n \in \mathbb{N}$, n is a proper subset of $n^+ = n \cup \{n\}$ and so $n \neq n^+$.

2) Prove by mathematical induction.

Order Relation of Natural Numbers.

Definition 2.3

Let $m, n \in \mathbb{N}$.

n is said to be greater than or equal to m if $m \subseteq n$, and we denote it by $m \leq n$.

n is said to be greater than m if $m \leq n$ and $m \neq n$, and we denote it by $m < n$.

Therefore, $m < n$ if m is a proper subset of n.

Immediate consequence from the above definition and properties of $\subseteq$.

Proposition 2.3

Let $m, n, p \in \mathbb{N}$

1) $0 \leq n$

2) $m \leq n$ and $n \leq m$ if and only if $m = n$.

3) if $m \leq n$ and $n \leq p$, then $m \leq p$.

Proposition 2.4 / Exercise 2.1

Let $m, n \in \mathbb{N}$

1) $m < n^+$ if and only if $m \leq n$.

2) $m^+ \leq n$ if and only if $m < n$.

Then $m < n \Leftrightarrow m^+ \leq n$, i.e. $m^+ = m \cup \{m\} \subseteq n \Leftrightarrow m \in n$

Therefore, for all $n \in \mathbb{N}$, $n = \{m \in \mathbb{N} : m < n\}$.

Let $m, n \in \mathbb{N}$. From the definition of "$<$", if $m = n$ is true, then $m < n$ and $n < m$ are false.

If $m \neq n$, from 2) of proposition 2.3, we have $m < n$ or $n < m$. Suppose that both $m < n$ and $n < m$ are true, $m < n \Rightarrow m^+ \leq n$ and $n < m \Rightarrow n^+ \leq m$. Therefore, $m \cup \{m\} = m^+ \subseteq n \subseteq n^+ \subseteq m \Rightarrow m \in m$ which contradicts to 3) of lemma 2.1.

As a result, $m = n$, $m < n$ and $n < m$ are exclusive cases!

Question: If m, n are given, can they always be compared?

Theorem 2.2 (Law of Trichotomy)

Let $m, n \in \mathbb{N}$, then exactly one of the following three statement is true:

(1) $m = n$, (2) $n < m$, (3) $m < n$.

proof:

Let S be the subset of $\mathbb{N}$ whose elements satisfy the above

(i) Let $m = 0$ and $n \in \mathbb{N}$.

If $n = 0$, then $m = n = 0$. Also $n < m$ and $m < n$ are false as both m, n are empty.

If $n \neq 0$ (i.e. $m = n$ is false), since $0 \leq n$ for all $n \in \mathbb{N}$, we have $0 < n$ (i.e $m < n$).

$\therefore 0 \in S$.

(ii) Suppose that $m \in S$. Let $n \in \mathbb{N}$.

Case 1: $n \leq m \Rightarrow n < m^+$ ((1) of proposition 2.4)

Case 2: $m < n \Rightarrow m^+ \leq n$ ((2) of proposition 2.4)

$\therefore m^+ \in S$.

$\therefore S = \mathbb{N}$.

Theorem 2.5 (Well-ordering property)

Every non-empty subset M of $\mathbb{N}$ contains a least element, i.e. there exists $n \in S$ such that for all $m \in M$, we have $n \leq m$.

proof.

Define $L = \{x \in \mathbb{N} : x \leq y \text{ for all } y \in M\}$

Note: 1) $0 \in L$, so $L \neq \phi$.

2) $M \neq \phi$, let $m \in M$, then $m < m^+$ and so $m^+ \notin L$, i.e. $L \neq \mathbb{N}$.

(Contrapositive of) 5) of theorem 2.1 $\Rightarrow$ there exists $s \in L$ but $s^+ \notin L$.

$s \in L \Rightarrow s \leq m$ for all $m \in M$

Claim: $s \in M$

Suppose not, then $s < y$ for all $y \in M$ and so $s^+ \leq m$ for all $y \in M$ which means $s^+ \in L$ (Contradiction).

Arithmetic of Natural Numbers

Definition 2.3 (Addition of Natural Numbers)

If $m, n \in \mathbb{N}$,

1) $m + 0 := m$

2) $m + n^{+} := (m+n)^{+}$.

Example 2.1

$1+1 = 1+0^{+} = (1+0)^{+} = 1^{+} = 2$

$3+2 = 3+1^{+} = (3+1)^{+} = (3+0^{+})^{+} = ((3+0)^{+})^{+} = (3^{+})^{+} = 4^{+} = 5$

Proposition 2.2

1) (Associativity) For all $m, n, p \in \mathbb{N}$, $(m+n)+p = m+(n+p)$

2) (Existence of Identity) For all $m \in \mathbb{N}$, $m+0 = 0+m = m$.

3) (Commutativity) For all $m, n \in \mathbb{N}$, $m+n = n+m$

proof:

1) Prove by induction on p.

① When $p = 0$,

$(m+n)+p = (m+n)+0 = m+n$

$m+(n+p) = m+(n+0) = m+n$

② Suppose that $(m+n)+p = m+(n+p)$,

$(m+n)+p^{+} = ((m+n)+p)^{+} = (m+(n+p))^{+} = m+(n+p)^{+} = m+(n+p^{+})$

2) $m+0 = m$ (Definition)

Prove that $0+m = m$ by induction on m.

① When $m = 0$,

$0+m = 0+0 = 0 = m$

② Suppose that $0+m = m$,

$0+m^{+} = (0+m)^{+} = m^{+}$

3) Prove the special case $m+1=1+m$ by induction on m.

① When $m=0$,

$$0+1=1+0$$

② Suppose that $m+1=1+m$

$$m^{+}+1=(m+0)^{+}+1=(m+0^{+})+1=(m+1)+1=(1+m)+1=1+(m+1)=1+(m+0^{+})=1+(m+0)^{+}=1+m^{+}$$

Prove the general case $m+n=n+m$ by induction on n.

① When $n=0$,

$$m+0=0+m \quad (\text{By } 2)$$

② Suppose that $m+n=n+m$,

$$m+n^{+}=m+(n+1)=(m+n)+1=(n+m)+1=1+(n+m)=(1+n)+m=(n+1)+m=n^{+}+m$$

Definition 2.3 (Multiplication of Natural Numbers)

If $m,n\in\mathbb{N}$,

1) $m\times 0:=0$

2) $m\times n^{+}:=m\times n+m$

Example 2.2

$4\times 1=4\times 0^{+}=4\times 0+4=0+4=4$

$4\times 2=4\times 1^{+}=4\times 1+4=4+4=8$

Proposition 2.3 / Exercise 2.1

1) (Associativity) For all $m,n,p\in\mathbb{N}$, $(m\times n)\times p=m\times(n\times p)$

2) (Existence of Identity) For all $m\in\mathbb{N}$, $m\times 1=1\times m=m$.

3) (Commutativity) For all $m,n\in\mathbb{N}$, $m\times n=n\times m$

4) (Distributivity) For all $m,n,p\in\mathbb{N}$, $(m+n)\times p=m\times p+n\times p$ and $p\times(m+n)=p\times m+p\times n$.